People and Events Behind the Fractal Images

When asked to write this article I, without space limitation, have unleashed a flood of recollections about some men and some ideas involved in the art of fractals, including both art for art's sake and art for the sake of science. A few of these recollections may even qualify as history, or perhaps only as what the French call *le petite histoire*. As some readers may already know, history for me is forever a part of the present.

The Prehistory of some fractals-to-be: Poincaré. Fricke & Klein, and Escher

To begin, while fractal geometry dates from 1975, it is in many ways important to know that a number of shapes now called fractals have been known for a much longer time. But surprisingly few had actually been drawn before the computer era. Most were self-similar or self-affine, and represent the artless work of the draftsmen on the payroll of science publishers. Also, there are renditions of physical and simulated Brownian motion in the book by Jean Perrin, *Les Atomes*, and William Feller's *Introduction to Probability*. These renditions have helped me dream in fruitful ways (as told in my 1982 book Figure *The Fractal Geometry of Nature* p. 240), but they are not beautiful. Fractals-to-be-occur in the work of Fatou and Julia circa 1918, but they led to no illustrations in their time.

However, Poincaré's even earlier works circa 1890 do include many sketches, and two very different nice stories are linked with illustrations that appeared shortly afterwards, in the classic book *Vorlesugen ✧ber die Theorie der automorphen Functionen*, Fricke & Klein 1897. This book's text and its preface are written by Robert Fricke, but we read on p. vi of the book that the great Felix Klein, "a teacher and dear friend," seems to have graciously consented to having his name added on the title page. The illustrations became even more famous than the text. They have been endlessly reproduced in books on mathematics, and, for better or worse, have affected the intuition of countless mathematicians.

A tenacious legend claims that students in industrial drawing at the Technische Hochule in Braunschweig, where Fricke was teaching mathematics, drew these figures as assignment, or perhaps even as an exam. Unking words have been written about some of the results.

In fact, I have done my share of detailing the defects of those figures which claim to represent the fractal-to-be limit sets of certain Kleinian groups (leading some to wonder which of Fricke's students should b failed posthumously). These two dubious figures were drawn with the help of the original algorithm of Poincaré, which is very slow, too slow even for a computer. However, my paper in *The Mathematical Intelligencer,* M 1983m, has given an explicit and quick new algorithm. The comparison is summarized in Figure *The Fractal Geometry of Nature*, page 179. As was to be expected, the actual shape is by far the more detailed and refined of the two. But this is not all: against all expectations, it is *not* necessarily perceived as more complicated. I feel it is more harmonious, and can be comprehended as a whole, therefore it is perceived as far simpler than the clumsy old pictures. However, a famous mathematician (15 years my senior) has expressed dismay at seeing the still vibrant foundation of his intuition knocked down by a mere machine.

Of wider popular interest by far are Fricke's drawings of "hyperbolic" tessellations," the reason being that they have become widely popular behind diverse embellishments of Morits C. Escher, as seen, for example, in the book Image *The World of M.C. Escher*. Many people immediately perceive some "obvious but hard to describe" connection between Escher and fractals, and it is good to know that these tessellations are indeed closely related to fractals.

In fact, they were knowingly triggered by Poincaré, as is well documented by H.S.M. Coxeter in his 1979 *Leonardo* paper. Having seen some of Escher's early work, this well-known geometer wrote to him and received the following answer: "Did I ever thank you ...? I was so pleased with this booklet and proud of the two reproductions of my plane patterns!... Though the text of your article [in *Trans. Royal Soc. Canada*, 1975] is much too learned for a simple, self-made plane pattern-man like me, some of the illustrations.... gave me quite a shock... Since a long time I am interested in patterns with "motives" getting smaller and smaller till they reach the limit of infinite smallness... but I was never able to in which each "blot" is getting smaller gradually from a centre towards the outside circle-limit, as [you] show... I tried to find out how this figure was geometrically constructed, but I succeeded only in finding the centers and radii of the largest inner-circles. If you could give me a simple explanation..., I should be immensely please and very thankful to you! Are there other systems besides this one to reach a circle limit? Nevertheless,... I used your model for a large woodcut." This was his picture "Circle Limit I," concerning which he wrote on another occasion: "This woodcut *Circle Limit I*, being a frist attempt, displays all sorts of shortcomings."

In his reply, Coxeter told Escher of the infinitely many patterns which tessellate a Euclidean or non-Euclidean plane by black and white triangles. Escher's sketch-boks show that he diligently pursued these ideas before completing *Circle limites II, III, and IV*. He wrote: "In the colored woodcut *Circle Limit III*, most of the defects [of *Circle limit I*] have been eliminated." In his *Magic Mirror of M.C. Escher* (1976), Bruno Ernst wrote: "best of the four is *Circle Limit III*, dated 1959... In addition to arcs placed at right angles to the circumference (as they ought to be), there are also some arcs that are not so placed." [Now going back to Coxeter] "In fact all white arcs 'ought' to cut the circumference at the same angle, namely 80 degrees (which they do, with remarkable accuracy). Thus Escher's work, based on his intuition, without any computation, is perfect, even though his poetic description of it was only approximate."

The reader is encouraged to read Coxeter's paper beyond these brief quotes. But an important lesson remains, and deserves to be restated. The Coxeter pictures, which made Escher adopt the style for which he became famous, hence eventually affected the aesthetics of many of out contemporaries, were not the pure creation of an artist's mind. They came straight from Frick & Klein, they were largely inspired by Henri Poincaré, and they belong to the same geometric universe as fractals. Also note that the preceding story is one of only two in this paper to involve a person who had been professionally trained as an artist.

The Fractal Mountains of R. F. Voss

My next and very pleasant task is to tell how I met Voss and some other people important to the story of the *Art of Fractals*.

During the Spring of 1975, Richard F. Voss was hopping across the USAA in search of the right job. He was soon to become Dr. Voss, on the basis of a Berkeley dissertation whose contents ranged from electronics to music, without ever leaving the study of a widespread physical phenomenon (totally baffling then, and almost equally baffling today), called *1/f noise*. Other aspects of this noise, all involving fractals, were favorite topics of mine since 1963, and my book Figure *Les objets fractals*, which was based on a generalization of 1/*f* noise from curves to surfaces. One of the more striking parts of Voss's thesis concerned (composed) music, which he discovered had many characteristics involving 1/*f* noises. He had even based a micro-cantata on the historical record of Nile River discharges, a topic dear to my heart.

Therefore, Voss and I spoke after his job-hunting talk at IBM, Yorktown, and I offered him a deal: come here and let us play together; something really nice is bound to come out. He did join the Yorktown low-temperature group and we soon became close co-workers and friends. Contrary to what is widely taken for granted, he never joined my tiny project, and he spent the bulk of his time on experimental physics. Nevertheless, his contribution to fractals came at a critical juncture, and it has been absolutely essential. First, we talked about writing a book on 1/*f* noise, but this project never took off (and no one else has carried it out, to my knowledge, to this day). Indeed, each time he dropped by, to try and do something together, he found me involved with something very different, translating and revising Figure *Les objects fractals*. The end product came out in 1977 as Figure *Fractals*. There were many graphics problems in its preparation. Voss ceaselessly inquired about what Sig Handelman and I were doing, and kept asking whether we would consider better ways. Then he found a sure way of obtaining our full attention: he conjured a computer graphics system where none was supposed to exist, and brought along pictures of fractals that were way above what we had been dealing with until then. They appeared in *Fractals*, which is why the foreword describe him as the co-author of the pictures in that book.

Color came late to Yorktown, where it seems we fractalists continued to be the only ones to use demanding graphics in our work. We first used color in my next book, the 1982 Figure *Fractal Geometry of Nature*. In late 1981, the text was already in the press, but the color pictures had not yet been delivered to the publishers. The film recorder we were using was ours only on a short lease, and this fact and everything else was conspiting to make us rush, but I fought back. Since 'the desire is boundless' (Figure FGN, p. 38), I fought hardest for the success of the *Fractal Planetrise* on the book's jacket. It was soon refined to the point of what (by the standards of the day) was perfection, but this was not enough. Just another day's work, or another week's, I pleaded, and we shall achieve something that would not need any further improvement, that would not have to be touched up again when the "lo-fi" graphics of the day were to be replaced by the "hi-fi" graphics. To my delight, Voss was also a perfectionist.

Fractal illustrations had started as wholly ulitarian; the perceived beauty of the old ones by Jean-Louis Oneto and Sig Handelman was an unexpected and unearned bonus. Buy by 1981 their beauty had matured and it deserved respect, even from us hard scientists, and it deserved time. Many people have, since those days, showed me their fractal pictures by the hundreds, but I would have been happier in most cases with fewer, more carefully worked out ones.

Everyone experiences wonder when observing Voss's pictures, and "to see [them] is to believe [in fractal geometry]." Specialists also wonder how these pictures were done, because, without ever drawing specific attention to the fact, Voss has repeatedly conjured technical tricks that were equivalent to computer graphics procedures that did not officially develop until much later. This brings to mind a philosophical remark.

Watching Voss the computer artist, and Voss the physicist at work for many years had kept reminding me of the need for a fruitful tension between the social and the private aspects of being a scientist. The only civilized way to be a scientist is to engage in *the process* of doing science primarily for one's *private pleasure*. To derive pleasure form the *public results* of this process is a much more common and entirely different matter. The well-known danger is that, while *dilettante*, is a term of contempt. While not a few individuals profess to be serious scientists, yet many are motivated primarily by personal enjoyment of their work, very few could provide what I view as the only acceptable evidence of "serious dilettantism." This demonstrates a willingness and, perhaps, even a compulsion to leave significant portions of one's best work unpublished or unheralded – knowing full well that one could claim no credit for them. This may be easiest for the scientific giants; lars Onsager was a legend in this respect. On the other hand, every scientist has been the active or the passive witness of episodes when one could not or would not work in a field without becoming thoroughly uncivilized. The true test, therefore, arises when civilized behavior is neither easy nor impossible. On these, and other stringent grounds, I view Dick Voss (as graphics expert and as physicist) as being one of the most civilized serious scientists in my wide acquaintance.

Old Films: Competing with the Good Lord on Sunday

What about films? We were ill-equipped to produce them, having only an exhausted film recorder (a tube-based Stromberg Carlson 4020) at our disposal. In 1972 with Hirsh Lewitan, however, we did prepare a clip on the creation of fractal galaxy clusters, using the *Seeded Universe* method. Then, in 1975, with Sig Handelman, we added a clip on fractal mountains: the landscape later used as Plate 271 of Figure *The Fractal Geometry of Nature* emerged slowly from the deep, then rotated majestically (or at least very slowly), and finally slipped back under water. Everyone spontaneously called this the *Flood* Sequence. By a fluke, the highest altitude achieved at two distinct points, and a programming flaw stopped the Flood when these points were still visible. Delighted, I indulged in commenting that my fractal model of relief had predicted that there were two tips to Mount Ararat, not one ... until an auditor straight from Armenia reported very dryly that this fact was well-known to everyone in his country.

The *Galaxy Clustering* and the *Flood* sequences, taken together, were nicknamed *Competing with the Good Lord on Sunday*. They soon came to look out-of-date and pitifully primitive, but now they look good again: they are of historical interest ... valuable antiques.

In the *Flood*, the observer simply moved around a landscape without zooming. The same was true in the animation of one of Voss's data bases, done by R. Greenberg Associates for an IBM commercial clip.

Star Trek II

But what of the “real Hollywood?” “It” immediately realized the potential of Voss’s landscape illustrations in my 1977 book, and soon introduced variants of these fractals into its films. This led to a lovely reenactment of the old and yet always new story of Beauty and the Best, since it is taken for granted that films are less about Brains than about Beauty, and since the few brainy mathematicians who had known about individual fractals-to-be, had taken for granted (until my books) that these were but Monsters. Beastly. The pillars of “our geographically extended Hollywood” were Alain Fournier, Don Fussell and Loren Carpenter. Early in 1980, John W. Van Ness, a co-author of mine in 1968 who had moved to the University of Texas at Dallas, asked me to comment on the draft of his student Fournier’s Ph.D. dissertation. Fournier and Fussell had written earlier to as, asking for the IBM programs to generate fractal mountains, but we did not want to deal with lawayers for the sake of programs that were not documents, and the programs were already too intimately linked to one set of computers to be readily transported anywhere else. Therefore, Fournier and Fussell went their own way, and soon hit upon an alternative method that promised computations drastically faster than those of Voss.

Precisely the same alternative was hit upon at the same time by Loren Carpenter, then at Boeing Aricraft, soon to move to Lucasfilm, and now at Pixar. In his own words in *The College Mathematics Journal* of March 1984, “I went out and bought [Figure *Fractals*] as soon as I read [Martin] Gardner’s original column on the subject in *Scientific American*. I have gone through it with a magnifying glass two or three times. I found that it was inspirational more than anything else. What I got out of it myself was the notion that ‘Hey, these things are all over, and if I can find a reasonable mathematical model for making pictures, I can make pictures of all the things fractals are found in.’ that is why I was quite excited about it...”

“The method I use is recursive subdivision, and it has a lot of advantages for the applications that we are dealing with here; that is, extreme perspective, dynamic motion, local control – if I want to put a house over here, I can do it. The subdivision porocess involves a recursive breaking-up of large triangles into smaller triangles. We can adjust the fineness of the precision that we use. For example, in ‘Star Trek,’ the images were not computed to as fine a resolution as possible because it is an animated sequence and things are going by quickly. You can see little triangles if you look carefully, but most people never saw them.”

“Mandelbrot and others who have studied these sorts of processes mathematically have long been aware that there are recursive approximations to them, but the idea of actually using recursive approximations to make pictures, a computer graphics-type application, as far as we know first occurred with me and Fournier and Fussel, in 1979...”

“One of the major problems with fractals in synthetic imagery is the control problem. They tend to get out of hand. They will go random all over the place on you. If you want to keep a good tight fist on them and make it look like what you want it to look like, it requires quite a bit of tinkering and experience to get it right. There are not many people around who know how to do it.”

While still at Boeing, Carpenter became famous in computer graphics circles for making a short fractal film, *Vol Libre*, and he was called to Lucasfilm to take a leading role in the preparation of the full feature Figure *Star Trek II: The Wrath of Khan*. Several computer-generated sequences of this film involve fractal landscapes, and have also become classics in the core computer graphics community. The best known is the *Genesis* planet transformation sequence. A different company, Digital Productions, later included massive fractal landscapes in *The Lost Starfighter*, which I saw – without hearing it – in an airplane. I had seen *Star Trek II* in a suburban movie-house (since I had gone there on duty, my stub was reimbursed). An associate had seen it on a previous day, and had reported that it was too bad that the fractal parts had been cut (adding as consolation that it was known that they *always* cut out the best parts in the suburbs). Of course, my wife and I immediately saw where the fractal portion started, and we marveled: If someone less durably immersed than the two of us in these matters could be fooled so easily, what about people at large?

Later, when he was interviewed for the summer 1985 issue of *La lettre de l'image*, Carpenter described the severe cost constraints imposed by his work: "We cannot afford to spend twice as much money to improve the quality of the pictures by 2%." One would hate to be asked to attach a numerical percentage to quality improvement, but computer costs do keep decreasing precipitously, and there is hope that future feature films using fractals will be affordable while pleasing even the crankiest mathematician.

This Beauty and the Beast episode was enjoyable but drew us into a few scrapes, long emptied of bitterness, but still instructive. we were disappointed that the endless credits of the films never included the word *fractal*, nor our names. Once excuse was that everyone who mattered knew, so there was no need to say anything. Besides, lawyers feared that, if mentioned, we would have been put in a position to sue for a part of the cake. The world at large does not believe that scientists are resigned tot eh fact that their best work – the principle of mathematics and the laws of nature – cannot be patented, copyrighted, or otherwise protected by law. All that the scientists can expect is to be paid in the coin of public – not private – praise.

Later on, we greeted with amusement Alvy Ray Smith's tern "graftal." The differences from "fractal" were hardly sufficient to justify this proprietary variation on my coinage.

Fournier & Fussel and Carpenter are not represented in *The Science of Fractal Images*. It is a pity that we did not come to know them better. They have hardly ever written to us, even at times when we could have helped, which we would have loved to do, and, in any case we would have linked to follow their work as it evolved.

Midpoint Displacement in Greek Geometry: the Archimedes construction for the Parabola

Our scrapes with "our Hollywood" have led to a variety of mutually contradictory impressions. Some people came to believe that the fractal landscapes in Fournier, Fussell & Carpenter 1982 are, somehow, not "true fractals." Of course *they are fractals*, just as true as the Koch curve itself. Other people believe that I begrudge credit for "recursive subdivision," in order to claim "midpoint displacement" – which is the same thing under a different term – for myself. Actually, as the French used to be taught in high school geometry, the basic credit for the procedure itself (but of course not for fractals) belongs to someone well beyond personal ambition, to Archimedes (287 – 212 BC).

The antiquity of the reference is a source of amusement and wonder, but rest assured that his work is amply documetned. A great achievement of Archimedes was when he evaluated the area between a parabola and a chord AB, an achievement that many writers view as the first documented step towards calculus. The technique Archimedes used was to take a chord's endpoints and interpolate recursively to values of x so that they form an increasingly tight dyadic grid. Using this, Archimede's was able to derive the rule of upward displacements though the parabola would not have an equation until Descartes devised analytic geometry.

Fractal Clouds

The algorithm Voss used to generate fractal mountains extends to clouds, as described in his contribution to this book. The resulting graphics are stunning, but actually do not provide an adequate fit to the real clouds in the sky. This is the conclusion we had to draw from the work of Shuan Lovejoy.

Lovejoy, then a meteorology student in the Physics department at McGill University in Montreal, wrote to me, enclosing a huge draft of his thesis. The first half, concerned with radar observation,. was not controversial and sufficed to fulfill all the requirements. But the second half, devoted to the task of injecting fractals into meteorology, was being subjected to very rough weather by some referees, and he was looking for help. My feeling was that this work showed very great promise, but needed time to "ripen." (I was reminded of my own Ph.D. thesis, which was hurried to completion in 1952; I had been in a rush to take a post-doctoral position, a reason that soon ceased to appear compelling.) Hence, my recommendation to Lovejoy was that he should first obtain his sheepskin on the basis of his non-controversial work, and then join me as a post-doctoral student. I argued that he must not leave in his publications too many points that the unavoidable unfriendly critics could latch on to.

I was very impressed by Shaun's area-perimeter diagram, drawn according to fractal precepts in my 1977 book, which suggested that the perimeters of the vertical projections of clouds (as seen from zenith, for example from a satellite) are of fractal dimension about 4/3. Lovejoy 1982, a paper that limited itself to this diagram and a detailed caption, immediately became famous. A second of many parts of Lobejoy's thesis required far more work and, finally, I pitched in. Our joint paper came out years later as M & Lovejoy 1985. The illustrations of clouds have ytet to be surpassed. ??? Figure ???? By then, Lovejoy had drifted away from me. He had grown impatient with my refusal to reopen old fights that had been won to an acceptable degree, and by my deliberate preference for seeking "soft acceptance," with controversy only when it is unavoidable, as opposed to "hard acceptance," with unforgiving victims.

Clouds seem to pose a severe challenge to landscape painters, but one has achieved fame for his prowess. His name was Salomon van Ruysdaë l (1602-1670), and he brings to mind a question and a story. The question is whether fractal geometry can help us to compare the clouds of Ruysdael with those of Mother Nature. Elizabeth Carter was an undergraduate in meteorology at the University of California at Los Angeles (UCLA), in the group of Professor George L. Siscoe. Her hobby is photographing clouds, and she had found a nice way of getting academic credit for it. The fractal dimension was estimated for many varied clouds' contours 9as seen from a newarly horizontal direction – not the same thing as Lovejoy's views form the zenith). The conclusion was that Nature's clouds' D's are far more tightly bunched. In hindsigh, the result was to be expected: the painter chose to paint clouds that are dramatic, yet not impossible, hence his clouds' D's are near Nature's maximum.

Fractal Trees

Before moving to nonlinear fractals, it seemed logical to me, as manager of a tiny fractals group, to perform a few preliminary tests without perturbing the on-going programs. This is how a Princeton senior, Peter Oppenheimer, came to work with us for a few weeks. He later wrote his senior thesis on fractals and, eventually, he moved to the New York Institute of Technology on Long Island, and became an expert on fractal botany. Today he has competition from Przemyslaw Prusinkiewicz.

Drawing nonrandom fractal trees is comparatively easy, and there are several in Figure *The Fractal Geometry of Nature*. Drawing random fractal trees that are not of unrealistic "sparseness" presents a major difficulty; branches must not overlap. Suppose that a random tree is to be constructed recursively. Once cannot add a branch, or even the tiniest twig, without considering the Euclidean neighborhood where the additions will be attached. However, points that are close by, according to Euclidean distance, may be far away according to the graph distance taken along the branches. Therefore, a random recursive construction of a tree, going from trunk to branches and on to twigs, is by necessity a global process. One may be drawn to seek a construction by self-contemplation, or by obeying the constraints imposed by one's computer's better way.

By contrast, space appears forgiving; more precisely, it offers an almost irresistible temptation to cheat. Show a shape described as a tree's projection on a plane, and challenge yourself to imagine a spatial tree having such a projection. Even when the original spatial branches happen to intersect or become entangled, our mind will readily disentangle them, and see them as a tree.

Now back to planar trees, and to ways of drawing them without worrying about self-intersection. A completely natural method was devised by Tom Witten and Leonard Sander. It came about in what we think is the best possible way, not during a search for special effects, but during a search for scientific understanding of certain web or tree-like natural fractal aggregates. The Witten-Sander method is called *diffusion limited aggregation*. Most unfortunately, it fails to yield realistic botanical trees, but it gives us hope for the future.

Iteration, Yesterday's Dry Mathematics, Today's Weird and Wonderful New Fractal Shapes, and the *Geometry Supercomputer Project*

Now, from fractals that imitate mountains, clouds, and trees, let us move on to fractals that do not. For the artist and the layman, they are simply weird and wonderful new shapes. My brief involvement with Poincaré limit sets has already been touched upon. My initiation to Julia sets began at age 20, when the few who knew them called the J-sets. This, and the beginning of my actual involvement with the study of iteration of $z \rightarrow z^2 + c$, have both been described in an invited contribution to Figure *The Beauty of Fractals*, and need not be repeated here.

But I do want to mention a brief interaction with David Mumford which eventually contributed to a very interesting and broad recent development.

David's name was known to me, and to everyone else in mathematics, because of his work in algebraic geometry. We met when I came to Harvard in 1979, and, in November 1979, he came to a seminar I gave. After the talk, which was on iteration, he rushed towards me: "On the basis of what you have said, you should also look into Kleinian groups; you might even find an explicit construction for their limit set." "Actually," I responded, "I already have a nice algorithm for an important special case. Please come to my office, and I shall show you."

David came over and saw the algorithm that was eventually published as M 1983i, as told earlier. The exact words of our conversation are, of course, lost, but I should remember their substance as follows: "This is so simple, that Poincaré should have seen it, or someone else singe Poincaré. Why did this discovery have to wait for you?" – "Because no one before me has used a powerful new tool, the computer!" – "But one cannot prove anything with a computer!" – "Sure, but playing with the computer is a source of conjectures, often most unexpected ones. The conjecture it has suggested about Kleinian limit sets has been easy to prove; other are too hard for me." – "In that case, would you help me learn to play with the computer?" – "With pleasure, but we would have to get help from my latest IBM assistant, Mark laff."

Soon afterwards, it became clear that Mumford had to seek associates closer by, in Cambridge. He was tutored by my course assistant Peter Moldave, and started working with David Wright, then a Harvard graduate student in mathematics, who ceased, at that point, to hide his exceptional programming skills. Eventually, Mumford became thoroughly immersed in computer, first as heuristic tools, then for their own sake.

He became instrumental in helping the awareness of the computer-as-tool idea that spread among mathematicians. The resulting needs grew so rapidly that, after barely eight years, the Figure National Science Foundation had established a *Geometry Supercomputer Project*! The charter members are F. Almgren (Princeton), J. Cannon (Brigham Young), D. Dobkin (Princeton), A. Douady (ENS, Paris), D. Epstein (Warwick), J. Hubbard (Cornell), B. Mandelbrot (IBM & Yale), A. Marden (Minnesota), J. Milnor (IAS, Princeton), D. Mumford (Harvard), R. Tarjan (Princeton & Bell Labs), and W. Thurston (Princeton). At the risk of sounding corny, let me confess that the opening of this project was a high point in my life. In 1991 it expanded, and its name changed to *Geometry Center*.

The next topic to be discussed concerning iteration is my fruitful interaction with V. Alan Norton, a Princeton mathematics Ph.D. in 1976, who was in my group as a post-doc in 1980-82, and stayed on the research staff at IBM Yorktown. He was one of the two principal "illustrators" of Figure *The Fractal Geometry of Nature*, as seen in that book's very detailed picture credits. He has achieved great renown, starting with Siggraph 1982, for splendid quaternionic Julia set pictures.

Norton also worked on the end-papers, without legend, for *The Fractal Geometry of Nature* Figure. How these end-papers came about is a tale worth recounting. They involve an important problem from the theory of iteration of analytic functions, an artifact due to inherent limitations of the computer, and two decorative touches.

The original graph was unbounded, and Norton introduced a decorative touch: inversion with respect to a circle. I loved the result; unfortunately, while bounded, it did not fit neatly on a double page spread. Hence I imposed a second and more arbitrary decorative touch: the horizontal stretching of the graph to fill the available space.

The serious mathematical problem that had led me to this graph was the use of Newton's method to solve the equation exp(z0) = c. The solutions are known from calculus, but Gaston Julia had shown in 1917 that Newton's method is a fruitful ground for the study of the iteration of functions of a complex variable z. Chapter 19 of *The Fractal Geometry of Nature* Figure examines the iteration of z^2 + c and other polynomials. This end-paper relates to the iteration of the transcendental function $z - 1 + Ce^{-z}$.

In Arthur Cayley's pioneering *global* studies of iteration in 1879, the interest in iteration had arisen from the application of the Newton-Raphson method. (Peitgen et al. tell the story, and illustrate it, in *The Mathematical Intelligencer* in 1984.) Cayley began by solving $z^2 = C$, which proved easy, then went on to try $z^3 = C$, which stumped him by exhibiting three "gray areas: that he found no way of resolving. Julia, in 1917, had found many acts about these areas, and John H. Hubbard had shown us his revealing earliest graph of the corresponding Julia set. It was natural for us, in late 1980, to play with $z^p = C$, and then view exp(z) = C as a suitable limit of $z^p = C$ for $p \to$inf. We made many interesting observations about this limit case, but the study was far form complete and publishable when we moved on to very different work.

Finally, and unfortunately, the non-fractal bold boundaries between the background and the solidly colored areas in the end-papers of *The Fractal Geometry of Nature* Figure are an artifact. The study of transcendental functions' iterates leads very quickly to enormous integers, hence soon reaches intrinsic limits beyond which the computer takes its own arbitrary actions.

Devaney, Barnsley, and the Bremen Book, *The Beauty of Fractals*

Out perception of the iteration of transcendental functions as a difficult and very rich topic was confirmed by several eminent mathematicians, such as Robert L. Devaney. No wonder, therefore, that one should see striking resemblances between our end-papers and his beautiful and widely seen illustrations and films. Bob's papers on the iteration of transcendental functions had already brought admiring attention to him, but we did not become fast friends until we started bumping into each other constantly on the fractal *Son et Lumière* traveling shows.

The life orbit of Michael barnsley has also crossed mine, and then they stayed in the same fractal neighborhood. The amusing background, in this instance, is in the public record, and I will not repeat it. I first read about it in James Gleick's book, *Chaos: The Birth of a New Science*. There I found out how it came to be that Michael burst into my house one day, full of enthusiasm, and lovely tales. Later, we held a few meetings at the Atlanta airport (of all places!), and since then it has been a pleasure to keep up with his work and that of his many associates.

Now back to the pictures of Julia and Mandelbrot sets in Figure *The Fractal Geometry of Nature*. During the summer of 1984, we were tooling up to redo them in color, with Eriko Hironaka as programmer, when mail brought in, hot off the press, the June issue of the German magazine *Geo.* We realized immediately that much of what we were proposing had already been achieved, in fact achieved beyond our aspirations, by

Heainz-Otto Peitgen, Peter H. Richter, Dietmar Saupe and their associates. Their fractal pictures in *The Mathematical Intelligencer* earlier in 1984 had been most welcome, but those in color had not yet supplied ample reason for enthusiasm. The color pictures in the 1984 *Geo* showed a skilled and artistic eye, coupled with a sure hand, one that had gained experience but had not become lazy or hasty. They were unmistakably the outcome of the search for perfection I had admired earlier in the work of Voss, and always attempt in my own.

I wrote to the *Geo* authors at the University of Bremen to congratulate them, to tell them of the change of plans their success had provoked, and to express the hope of meeting them soon. They told me about a fractal exhibit they were planning, and for which they were preparing a catalogue that eventually led to their book Figure *The Beauty of Fractals*, and they asked me to write the personal contribution mentioned earlier in this paper. That this book was fated not to come from me or my associates, it is still a delight that it came from them. I gained these new friends when they invited me to Bremen in may 1985, to open the first showing of their exhibit, and I participated in this exhibit in several other cities as well. Our joint appearances have since then have become too numerous to count. There are no anecdotes to tell, only very pleasant events to remember.

Conclusion

I am reminded that, only a while ago (the sting has disappeared, but the memory remains), no one wanted to scan my early pictures for longer than a few minutes, and this would-be leader of a new trend had not a single follower. Then, very slowly in memory, yet almost overnight in present perception, fractals became of so wide interest that Siggraph started devoting lectures, then full day courses to them. The first makeshift appearance of fractals at Siggraph came in 1985 under my direction, the second in 1986 under Peter Oppenheimer, and the third in 1987 and led to *The Beauty of Fractals* and this paper.