Math 152 (TTH)
Spring 2013
Exam 2
Mar 19, 2013
Time Limit: 50 Minutes

Name (Print): ____________________

Student number ____________________

This exam contains 12 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

The maximum grade you can get is 25. You may use the table to the right to get an idea of the weight of each question, but please don't write in it.

Problem	Points	Score
1	8	
2	7	
3	6	
4	4	
Total:	25	

The following table may be useful.

θ	$0(0^\circ)$	$\pi/6(30^\circ)$	$\pi/4(45^\circ)$	$\pi/3(60^\circ)$	$\pi/2(90^\circ)$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

I. Short Answer Problems

In problems (A)–(C), let $z = -2 + i$ and $u = 3 - 3i$.

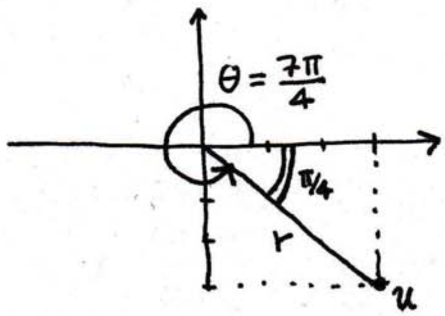
(A) (1 point) Calculate the value of $|zu|$.

$$|zu| = |z| \cdot |u| = \sqrt{(-2)^2 + 1^2} \cdot \sqrt{3^2 + (-3)^2}$$
$$= \sqrt{5} \cdot \sqrt{18} = \sqrt{90}$$

(B) (1 point) Calculate $w = \dfrac{u}{z}$ (i.e., express w as $w = a+ib$ where a, b are real numbers).

$$w = \frac{u}{z} = \frac{3-3i}{-2+i} = \frac{3-3i}{-2+i} \cdot \frac{-2-i}{-2-i}$$
$$= \frac{-6-3i+6i-3}{4+1} = \frac{-9+3i}{5}$$
$$= -\frac{9}{5} + \frac{3}{5} i$$

(C) (1 point) Express u in polar form (i.e., find $r > 0$ and θ such that $u = re^{i\theta}$).



$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18}$$

$$\theta = \frac{7\pi}{4}$$

$$u = \sqrt{18} \cdot e^{\frac{7\pi}{4} i}$$

(D) (1 point) Suppose we run the following MATLAB code.

```
A=ones(3,4);
for j=1:3;
  A(j,:)=j*A(j,:);
end;
```

Specify the output if you now type:

```
>> A(3,2)
```

3.

Reason:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & \boxed{3} & 3 & 3 \end{bmatrix}$$

A(3,2).

(E) (1 point) Let $A = \begin{bmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \end{bmatrix}$ be a 3×3 matrix with rows $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ (where each $\mathbf{a_j}$ is a 1×3 row vector). Suppose that $\det A = 3$. Find $\det \begin{bmatrix} \mathbf{a_3} \\ 2\mathbf{a_2} \\ \mathbf{a_1} \end{bmatrix}$.

$$\det\begin{bmatrix} a_3 \\ 2a_2 \\ a_1 \end{bmatrix} = 2 \det\begin{bmatrix} a_3 \\ a_2 \\ a_1 \end{bmatrix} = -2 \det\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = -2 \cdot \det A = -2 \cdot 3 = -6.$$

(F) (1 point) Is it possible for a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ to transform three *linearly dependent* vectors u, v and w into three *linearly independent* vectors $T(u)$, $T(v)$ and $T(w)$? If yes, give an example. If no, explain.

No, $T(u)$, $T(v)$ and $T(w)$ are again linearly dependent.

Since u, v and w are linearly dependent we can find scalars r, s and t so that $ru + sv + tw = 0$ and so that at least one of r, s and t is not zero.

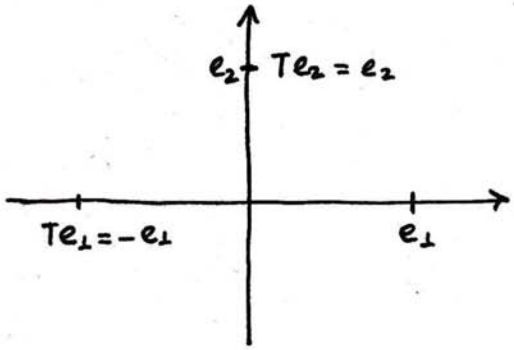
If you apply T to both sides of the previous equality we get

$$T(ru + sv + tw) = T(0)$$

$$r \cdot T(u) + s \cdot T(v) + t \cdot T(w) = 0. \quad \left(\begin{array}{c} T(0) = 0 \text{ for any} \\ \text{linear transformation} \end{array}\right).$$

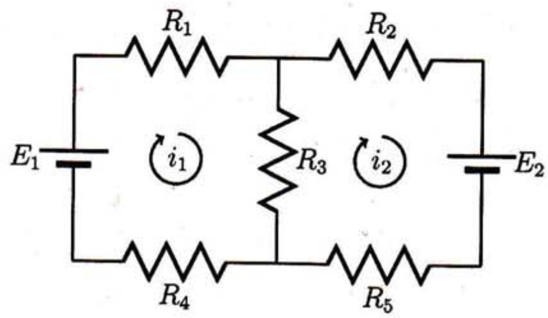
and since at least one of r, s and t is not zero, we conclude that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent as well.

(G) (1 point) Write the matrix representing the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that reflects a vector $\vec{x}$ about the vertical axis.



$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(H) (1 point) Consider the following circuit with $R_1 = 1\Omega$, $R_2 = 2\Omega$, $R_3 = 3\Omega$, $R_4 = 4\Omega$, $R_5 = 5\Omega$, $E_1 = 10V$ and $E_2 = 20V$.



Write an equation that describes the sum of voltage drops around loop 1 (i.e., the loop with current i_1).

$$R_1 i_1 + R_3 (i_1 - i_2) + R_4 \cdot i_1 - E_1 = 0$$

$$i_1 + 3(i_1 - i_2) + 4 i_1 = 10$$

$$8 i_1 - 3 i_2 = 10$$

II. Long Answer Problems

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

(a) (3 points) Find A^{-1}.

(b) (1 point) Find $\det(A^T)$.

(c) (1 point) Consider the equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. Write the corresponding linear system of equations.

(d) (2 points) Solve this linear system using (a).

(a) $$\left[\begin{array}{ccc|ccc} 2 & 4 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 5 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 5 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right].$$

Then: $A^{-1} = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

(b) $\det A^T = \det A = 2 \cdot \det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - 4 \det \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + 5 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = 0 + 4 - 5 = -1.$

(c) $$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + 4x_2 + 5x_3 = 1 \\ x_2 + x_3 = 1 \\ x_1 + 2x_2 + 2x_3 = 0 \end{cases}$$

(d) $$Ax = b \Rightarrow \underbrace{A^{-1}A}_{I}x = A^{-1}b \Rightarrow x = A^{-1}b, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

3. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2 + 2x_3 \\ kx_1 + x_2 \\ 2kx_1 + (k+2)x_2 + 4x_3 \end{bmatrix}$$

where k is a parameter.

(a) (2 points) Find the matrix A that represents the linear transformation T.

(b) (2 points) Calculate the determinant of A (as a function of the parameter k).

(c) (2 points) Determine all values of k for which the matrix A invertible?

(a)

$$A = \begin{bmatrix} 0 & 1 & 2 \\ k & 1 & 0 \\ 2k & k+2 & 4 \end{bmatrix}$$

(b)

$$\det A = 0 \cdot \det\begin{bmatrix} 1 & 0 \\ k+2 & 4 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} k & 0 \\ 2k & 4 \end{bmatrix} + 2 \cdot \det\begin{bmatrix} k & 1 \\ 2k & k+2 \end{bmatrix}$$

$$= 0 - 4k + 2\left[k(k+2) - 2k\right] = -4k + 2\left(k^2 + 2k - 2k\right)$$

$$= 2k^2 - 4k.$$

(c) A is invertible $\Leftrightarrow$ $\det A \neq 0$.

Then we solve

$$\det A = 0$$

$$2k^2 - 4k = 0$$

$$2k(k-2) = 0 \Rightarrow k = 0 \text{ or } k = 2.$$

Then A is invertible for all values of k except $k=0$ and $k=2$.

4. Let T be the linear transformation acting on vectors in $\mathbb{R}^2$ by first rotating the vector by $\pi/3$ (counterclockwise) and then reflecting the resulting vector across the vertical axis.
   (a) (2 points) Find the matrix for T.
   (b) (1 point) Find the matrix for T^{-1}.
   (c) (1 point) Find a vector $\vec{x}$ such that $T(\vec{x}) = \vec{x}$.

(a) Rotation: $\theta = \pi/3$

$$\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Reflection across vert. axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

(b) $$T^{-1} = \frac{1}{(-\frac{1}{2})(\frac{1}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})} \cdot \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

(c) $$T\vec{x} = \vec{x} \Rightarrow T\vec{x} = I\vec{x} \Rightarrow (T - I)\vec{x} = 0 \Rightarrow \begin{bmatrix} -3/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -3/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -3/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\sqrt{3}/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 - \frac{\sqrt{3}}{3}x_2 = 0 \Rightarrow x_1 = \frac{\sqrt{3}}{3}x_2. \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3\, x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3 \\ 1 \end{bmatrix} x_2,$$ where x_2 is a parameter.

Then, a vector $\vec{x}$ such that $T(\vec{x}) = \vec{x}$ is $\vec{x} = \begin{bmatrix} \sqrt{3}/3 \\ 1 \end{bmatrix}$.

(Rmk: Any multiple of $\begin{bmatrix} \sqrt{3}/3 \\ 1 \end{bmatrix}$ satisfies $T(\vec{x}) = \vec{x}$).