Math 152 (TTH)
Spring 2013
Exam 1
Feb 7, 2013
Time Limit: 50 Minutes

Name (Print): ____________________

Student number ____________________

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	8	
2	5	
3	3	
4	5	
5	4	
Total:	25	

The maximum grade you can get is 25. You may use the table to the right to get an idea of the weight of each question, but please don't write in it.

I. Short Answer Problems

In problems (A)–(D), let $\vec{u} = [1, -2, 3], \vec{v} = [3, 4, 0]$ and $\vec{w} = [-1, 5, 2]$.

(A) (1 point) Calculate $3\vec{u} - 2\vec{w}$.

$$3\vec{u} - 2\vec{w} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} - \begin{bmatrix} -2 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -16 \\ 5 \end{bmatrix}$$

(B) (1 point) Calculate $\|\vec{v}\|$.

$$\|\vec{v}\| = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5$$

(C) (1 point) Calculate $\vec{u} \times \vec{v}$.

$$\vec{u} \times \vec{v} = \det\begin{bmatrix} i & j & k \\ 1 & -2 & 3 \\ 3 & 4 & 0 \end{bmatrix} = i \cdot \det\begin{bmatrix} -2 & 3 \\ 4 & 0 \end{bmatrix} - j \cdot \det\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} + k \cdot \det\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$= -12i + 9j + 10k$$

$$= \begin{bmatrix} -12 \\ 9 \\ 10 \end{bmatrix}$$

(D) (1 point) Are the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ linearly independent?

We test whether the associated determinant is equal to zero:

$$\det\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix} = \det\begin{bmatrix} 1 & 3 & -1 \\ -2 & 4 & 5 \\ 3 & 0 & 2 \end{bmatrix} = 8 + 45 + 12 + 12 = 77 \neq 0.$$

Then $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent.

In problems (E)–(F), let $\vec{a}, \vec{b}$ and $\vec{c}$ be **non-zero vectors** in $\mathbb{R}^3$. For each statement, determine whether it is **true** or **false**. If true, give a brief explanation. If false, give a specific example for which the statement does not hold.

(E) (1 point) If $\vec{a} \cdot \vec{b} = 0$ then $\vec{a} \times \vec{b} \neq \vec{0}$.

True: $\vec{a} \times \vec{b} \neq 0$.

Since $\vec{a} \cdot \vec{b} = 0$ we deduce that the nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular. The only way to have $\vec{a} \times \vec{b} = 0$ is that $\vec{a}$ and $\vec{b}$ were parallel vectors, which cannot happen because they are nonzero and perpendicular. Then $\vec{a} \times \vec{b} \neq 0$.

(F) (1 point) If $\{\vec{a}, \vec{b}\}$, $\{\vec{b}, \vec{c}\}$, and $\{\vec{a}, \vec{c}\}$ are each linearly dependent, then $\{\vec{a}, \vec{b}, \vec{c}\}$ is also linearly dependent.

True: $\{\vec{a}, \vec{b}, \vec{c}\}$ is also linearly dependent.

Since $\{\vec{a}, \vec{b}\}$ is linearly dependent there is a linear combination $r\vec{a} + s\vec{b} = \vec{0}$ that is zero but at least one of the coefficients r, s is not zero.

We then have that $r \cdot \vec{a} + s \cdot \vec{b} + 0 \cdot \vec{c} = \vec{0}$ and not all coefficients used are zero.

Then $\{\vec{a}, \vec{b}, \vec{c}\}$ is also linearly dependent.

(G) (1 point) Consider the planes $3x + 2y - z = 5$ and $ax - 6y + bz = 2$. For what values of a and b are the planes parallel to each other?

If the planes are parallel then their normal vectors $n_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ and $n_2 = \begin{bmatrix} a \\ -6 \\ b \end{bmatrix}$ are parallel.

Since $n_1 \neq 0$, we can find a scalar λ so that $n_2 = \lambda n_1$.

Then we get $\begin{bmatrix} 3\lambda \\ 2\lambda \\ -\lambda \end{bmatrix} = \begin{bmatrix} a \\ -6 \\ b \end{bmatrix}$. This gives $a = 3\lambda$, $2\lambda = -6$ and $b = -\lambda$. Therefore $\lambda = -3$ and the values of a and b are $a = -9$, $b = 3$

so that the planes are parallel.

(H) (1 point) Write a MATLAB for loop that produces the following 10×20 matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & \cdots & 21 \\ 3 & 4 & 5 & 6 & \cdots & 22 \\ \vdots & & & & & \vdots \\ 11 & 12 & 13 & 14 & \cdots & 30 \end{bmatrix}$$

```
For i = 1:10
  For j = 1:20
    A(i,j) = i+j ;
  end;
end;
```

II. Long Answer Problems

2. Consider the vectors $\vec{v} = [1, -2, 3]$ and $\vec{u} = [2, 1, 5]$.

   (a) (3 points) Find a parametric description of the line through the point $(-1, 1, 3)$ orthogonal to both vectors $\vec{v}$ and $\vec{u}$.

$$\vec{v} \times \vec{u} = \det \begin{bmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix} = i \cdot \det \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} - j \det \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} + k \det \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= -13i + j + 5k = \begin{bmatrix} -13 \\ 1 \\ 5 \end{bmatrix}$$

This vector is parallel to the desired line.

The line in parametric form is:

$$\underbrace{\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}}_{\text{point in the line}} + t \cdot \underbrace{\begin{bmatrix} -13 \\ 1 \\ 5 \end{bmatrix}}_{\text{vector parallel to the line}}$$

where t is a parameter.

   (b) (2 points) Find the equation of a plane through the point $(-1, 1, 3)$ orthogonal to the vector $\vec{u}$.

Since the plane is orthogonal to $\vec{u}$ we can write its equation as:

$$2x + y + 5z = d$$

for some constant d. We find the value of d so that $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ is in the plane.

$$d = 2(-1) + (1) + 5 \cdot (3) = 14.$$

Therefore, the equation of the plane is

$$2x + y + 5z = 14.$$

3. (3 points) Consider the two planes given by the equations

$$2x + y + z = 6$$

and

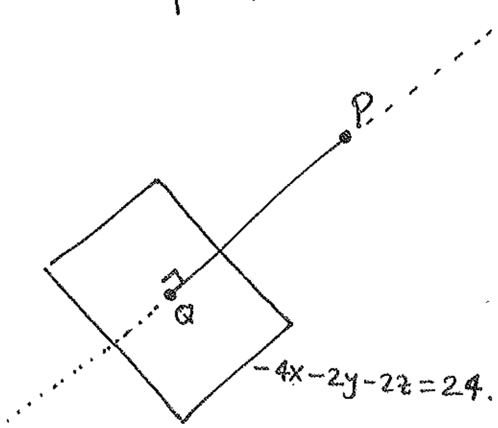
$$-4x - 2y - 2z = 24.$$

Find the distance between these planes.

The first plane has normal vector $n_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and the second plane has normal vector $n_2 = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$. Since these two normal vectors are parallel, then the two given planes are parallel.

Then, the distance between the two planes can be found as the distance from any point in the first plane to the second plane.

A point in the first plane is $P = \begin{bmatrix} 0 \\ 0 \\ +6 \end{bmatrix}$.



We find the point Q in the second plane so that $\overrightarrow{PQ}$ is perpendicular to the second plane.

The line from P to Q in parametric form is given by:

This line is Not needed.

$$\left(P + t\cdot\overrightarrow{PQ} = P + t(Q-P) = (1-t)P + tQ\right)$$

$$P + t\cdot n_1 = \begin{bmatrix} 0 \\ 0 \\ +6 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ t+6 \end{bmatrix}.$$

Now we find the value of the parameter t for which the corresponding point is in the second plane (that will give us the point Q):

$$-4\cdot(2t) - 2\cdot(t) - 2(t+6) = 24 \Rightarrow -12t = 36 \Rightarrow t = -3.$$

Then $Q = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + (-3)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 3 \end{bmatrix}$. Finally, the distance between the planes is equal to the distance between P and Q, which is:

$$dist(P,Q) = \|\overrightarrow{PQ}\| = \|Q-P\| = \left\|\begin{bmatrix} -6 \\ -3 \\ -3 \end{bmatrix}\right\| = \sqrt{(-6)^2+(-3)^2+(-3)^2} = \sqrt{54}.$$

4. (5 points) Find the value of h so that $\vec{d} = [1, -6, 4, h]$ is a linear combination of $\vec{a} = [1, -2, 1, -5]$, $\vec{b} = [0, -4, -1, 1]$, and $\vec{c} = [0, 0, 2, 3]$.

We need to find h so that the system $x \cdot \vec{a} + y\vec{b} + z\vec{c} = \vec{d}$ has solutions. In matrix form this system is

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & -4 & 0 \\ 1 & -1 & 2 \\ -5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 4 \\ h \end{bmatrix}.$$

To solve the problem we attemp to find the solution of this system by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -2 & -4 & 0 & -6 \\ 1 & -1 & 2 & 4 \\ -5 & 1 & 3 & h \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 3 & h+5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 3 & h+5 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & h+4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & h+4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & h-2 \end{array}\right]$$

From the last row we see that the system has solution exactly when $h-2=0$. That is $h=2$.

<u>Conclusion</u>: The vector $\vec{d}$ is a linear combination of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ if and only if $h=2$.

5. (4 points) The reduced row echelon form of the augmented matrix for a linear system is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right].$$

(a) What is the rank of this augmented matrix?

(b) Specify whether this linear system has a unique solution, infinitely many solutions, or no solution. Justify your answer.

(c) Find all solutions (if any) of the **corresponding homogeneous** system.

(a) The augmented matrix has four nonzero rows and it is in row echelon form. Then the rank of this augmented matrix is four.

(b) This linear system has no solution because the fourth equation is "$0=3$".

(c) For the corresponding homogeneous system we get the following reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we call the variables x_1, x_2, x_3, x_4 and x_5. Here x_1, x_2 and x_5 are basic variables, and x_3 and x_4 are free variables or parameters.

We get:

$$x_1 = 0$$
$$x_2 + 2x_3 + 3x_4 = 0$$
$$x_5 = 0$$

then:

$$x_1 = 0$$
$$x_2 = -2x_3 - 3x_4$$
$$x_5 = 0$$

Therefore the general solution to the homogeneous system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_3-3x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where x_3 and x_4 are free parameters.