Math 152 – Learning Goals

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1. Course-level Learning Goals

After completing this course, students should be able to

1. recognize word problems that can be formulated as linear systems, formulate such problems as linear systems, solve and/or analyze such systems (by hand in low dimensions, with computer assistance in higher dimensions) and relate any solutions and/or analysis back to the word problem. Also, interpret in this same manner the specific (and more technical) cases of resistor networks and random walks.

2. describe the connection between
   - linear geometric objects (vectors in co-ordinate space, lines and planes in 2D and 3D) and their basic properties (orthogonality, angles, projections, intersections) and
   - analytic quantities (vectors as column n-tuples, matrices, linear systems of equations) and their properties (determinants, dot and cross products, eigenvalues, etc.)
as well as relating computation in the latter domain to the geometry of the former.

3. recognize that linear systems of equations can have unique, infinite or no solutions, determine which of these situations applies, and describe the set of solutions (if any exist).

4. recognize matrix multiplication as a linear transformation and that such transformations (to the same dimensional space) can be simplified using eigen-analysis.

5. perform basic arithmetic with complex numbers, and thus extend eigen-analysis of matrices and linear systems to the complex case.

After completing the computer labs, students should be able to

1. operate and program in MATLAB (a computational tool that is commonly used in later courses and scientific/engineering careers) at a basic level: entering and operating on vector variables and matrix variables, solving linear systems, and implementing loops.

2. solve an applied word problem using MATLAB computations given the associated linear system, where the problem may be impractical to solve by hand (e.g., due to large dimension or large number of matrix multiplications); in the cases of resistor networks and random walks, be able to extract the associated linear system from the word problem and solve.

2. Topic-level Learning Goals

2.1 Vectors and Geometry

Topics:
1. Vectors: coordinate representation, length, dot product
2. Projection
3. 2x2 and 3x3 determinants
4. Cross product
5. Lines and planes in 2D and 3D
6. Geometry of solutions of linear systems
7. Linear dependence and independence

*Learning Goals: **After completing this section, students should be able to***
- represent quantities that have a magnitude and a direction as vectors.
- given a choice of coordinate axes, represent a vector as a column or a row using vector components.
- represent the standard basis vectors in 2D and 3D both graphically and as column vectors
- perform vector addition and scalar multiplication.
- compute the length of a vector given its components.
- define unit vector and compute the unit vector in the direction of a given vector.
- know the definition and basic properties of the dot product of two vectors.
- compute the dot product of two vectors.
- compute the angle between two vectors.
- determine when two vectors are orthogonal.
- find the projection of a given vector onto another vector.
- express a 2D or 3D vector using the standard basis vectors.
- define what a matrix is.
- compute determinants by hand in two and three dimensions.
- explain the relationship between the rows of a 2x2 determinant and the value of the determinant.
- know the definition and basic properties of the cross product of two vectors.
- calculate the area of a parallelogram spanned by two vectors (need not be 2D) by calculating the length of their cross product.
- given three vectors, compute their triple product (a 3x3 determinant calculation).
- calculate the volume of the parallelepiped spanned by the vectors via the triple product.
- determine whether the result of a vector calculation (dot, cross, triple products) is scalar or vector based on the expected result and calculations involved.
- find the parametric form and the equation form of points, lines (2D and 3D) and planes (3D) and relate the number of equations or parameters to the dimension of the object.
- determine whether a point is on a line/plane.
- find the point of intersection of lines and planes.
- determine if planes are parallel or find the their line of intersection.
- use either the parametric or the equation descriptions of lines and planes to solve simple geometrical problems (e.g., determine the line perpendicular to a certain plane passing through a certain point).

- provide a geometrical representation of a system of linear equations and its set of solutions in 2D and 3D.
- recognize a linear combination of vectors.
- give a parametric description of two and three dimensional space.
- discuss the difference between linearly dependent and independent vectors.
- give a geometrical interpretation of linear dependence in two and three dimensions.
- define a basis.

*Computer Lab Learning Goals: **Using MATLAB, students should be able to***
- enter vectors and matrices.
- perform component-wise operations on matrices using operators +, .* etc.
- use scalar functions sqrt, sin, cos, tan, asin, acos, atan, atan2 etc.
- use vector functions dot, cross, sum, norm etc.
- use matrix functions zeros and ones.

2.2 Systems of Linear Equations and Gaussian Elimination

Topics:
1. Solving linear systems
2. Elementary row operations
3. Gaussian elimination
4. Homogeneous equations and the structure of solutions
5. Geometric applications
6. Application to resistor networks

*Learning Goals: **After completing this section, students should be able to***
- given a system of linear equations in n unknowns, perform elementary row operations (Gaussian elimination) on the augmented matrix to transform the system into an easily solved system and find the set of solutions.
- recognize when a matrix is in row echelon form or reduced row echelon form and what operations are needed for augmented matrices in these forms to determine the solutions.
- determine when a linear system has a solution; when appropriate, write the set of solutions in parametric form and give a geometrical interpretation of that set.
- compute the rank of a matrix.
- discuss the set of solutions of a linear system based on the rank of the corresponding augmented matrix (i.e., determine the dimension and, when appropriate, the shape).
- discuss the set of solutions of an homogenous system of equations based on the number of unknowns and non-zero equations in the system (dimension and shape as above).
- relate the solutions of homogeneous to inhomogeneous systems.
- determine whether a set of vectors is linearly dependent or linearly independent.
- find the point or line of intersection of planes by solving a system of linear equations.
- write down a linear system describing the relations between currents and/or voltages in a resistor network, determine the set of solutions to the system, and form conclusions about the circuit dynamics based on these solutions.

Computer Lab Learning Goals: **Using MATLAB, students should be able to**
- extract a sub-matrix from a matrix or a particular entry in a vector or matrix.
- find the reduced row echelon form of an arbitrary matrix using the command rref.
- interpret the solution or solutions (if any) to a linear system by reading the result of rref on an augmented matrix.
- find the solution to a given system describing Kirchhoff's laws for a resistor network
- compute the voltage-to-current map matrix.

2.3 Matrices and Determinants

Topics:
1. Matrix multiplication
2. Linear transformations
3. Rotations, projections and reflections in 2D
4. Matrix representation and composition of linear transformations
5. Application to random walks
6. Matrix transpose
7. Matrix inverse
8. Determinants

Learning Goals: **After completing this section, students should be able to**
- determine the dimension of the result in matrix algebra calculations (scalar, vector of correct length, or matrix of correct size).
- perform matrix multiplication by hand in low-dimensions (i.e., usually lower than 5, unless the matrices have some special simple structure or are quite sparse).
- recognize "matrix times vector" multiplication as a linear combination of matrix column vectors.
- know what a transformation (mapping) is.
- determine when a transformation is linear.
- compute the resulting vector when a linear transformation is applied to a vector by using linearity and applying the transformation to basis vectors.
- recognize rotation, projection, and reflection in 2D as linear transformations of vectors, and provide simple examples of nonlinear transformations (via geometric descriptions, e.g. a "twist").
- interpret linear transformations as multiplication by a matrix, and find that matrix given how the transformation acts on a set of basis vectors.
- compose linear transformations and express the composition in terms of a matrix product.
- model random walks using vectors and matrices; use matrix multiplication to predict the probability that a system undergoing a random walk will be in a certain state at a specified future time given the system's initial state.
- state the definition and properties of the transpose of a matrix.
- state the definition of inverse of a matrix and determine when a matrix is invertible.
- compute the inverse of a matrix by hand in low-dimension (2D or 3D, unless the matrix has some special simple structure or is quite sparse).
- compute the determinant of an n x n matrix.

- compute the determinant of a triangular matrix.
- use row operations to aid in computing determinants.
- know about the linearity of determinants.
- identify matrix invertibility via a determinant calculation.
- apply the product formula for determinants.
- compute the determinant of the transpose.
- apply Cramer's rule for 2x2 systems.

Computer Lab Learning Goals: ***Using MATLAB, students should be able to***
- use the operator * to multiply matrices.
- analyze the behaviour of random walks that are large or where the long-term behaviour is of interest.
- find a solution to a linear system using the \ operator.
- use the function eye (identity matrix).
- compute the inverse of a matrix using the inv command.
- compute the transpose of a matrix using the ' command.
- compute the determinant of a matrix using the det command.
- create, edit, and run MATLAB scripts (not necessarily functions) using m-files.
- write scripts involving for loops that iterate a certain calculation or build a matrix with a given regular structure.

2.4 Eigenvalues and Eigenvectors

Topics:
1. Eigenvalues and eigenvectors
2. Complex numbers
3. Complex eigenvalues and eigenvectors
4. Diagonalization, powers of a matrix
5. Application to random walks
6. Complex exponential
7. Systems of linear differential equations
8. Vector differential equations

Learning Goals: ***After completing this section, students should be able to***
- define eigenvalues and eigenvectors of a matrix and discuss their geometric interpretation in terms of linear transformations (primarily in 2D).
- compute eigenvalues by calculating the zeros of the characteristic polynomial and find eigenvectors by determining the non-zero solutions of the corresponding system.
- find a basis of eigenvectors when it exists.
- interpret complex numbers as points on a plane and identify real and imaginary part.
- compute the modulus and the complex conjugate of a complex number.
- perform basic algebra with complex numbers: addition, multiplication, fraction simplification.
- find complex eigenvalues and eigenvectors when the characteristic polynomial has complex roots.

- relate the complex exponential to the unit circle in the complex plane.
- add and multiply complex exponentials.
- establish relationships between the eigenvalues of a diagonal matrix, the entries of that matrix and its determinant.
- perform matrix diagonalization using eigenvalues and eigenvectors.
- compute high powers of a diagonalizable matrix and calculate the determinant of a matrix, both by use of eigenvalues.
- explore the time evolution of a random walk using eigenvalues and eigenvectors.
- convert a system of differential equations into a vector differential equation and find the general solution to such an equation.
- find the solution of an initial value problem for a system of differential equations with real solutions.

Computer Lab Learning Goals: ***Using MATLAB, students should be able to***
- find eigenvalues and eigenvectors using the eig command.