Math 152 — Second Midterm

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1. **Do not open this exam until you are told to do so.**
2. **SPECIAL INSTRUCTIONS:** No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.
3. **Turn off all cell phones and pagers**, and remove all headphones.
4. This exam has 8 pages including this cover. There are 5 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
6. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
7. Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action. (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners. (b) Speaking or communicating with other candidates. (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
8. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam. You must not take any examination material from the examination room without permission of the invigilator.
9. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem	Points	Score
1	26	
2	16	
3	18	
4	20	
5	20	
Total	100	

1. [26 points] **Matrices and Determinants**

a. [5 points] Find the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 5 \\ -1 & 7 & -2 \end{bmatrix}$.

Solution:

$$\det A = 1 \cdot 6 \cdot (-2) + 2 \cdot 5 \cdot (-1) + 3 \cdot 7 \cdot 1 - 1 \cdot 6 \cdot (-1) - 2 \cdot 3 \cdot (-2) - 5 \cdot 7 \cdot 1 = -18.$$

b. [8 points] Find the inverse of the invertible matrix $D = \begin{bmatrix} 0 & -3 & 0 \\ -1 & 2 & 1 \\ 2 & -4 & -1 \end{bmatrix}$.

Solution: To find the inverse matrix we aument our given matrix by the identity on the right and we perform elementary row operations until we obtain the identity on the left:

$$\left[\begin{array}{ccc|ccc} 0 & -3 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -4 & -1 & 0 & 0 & 1 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 1 & 1 \\ 0 & 1 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

Therefore:

$$A^{-1} = \begin{bmatrix} -2/3 & 1 & 1 \\ -1/3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

c. [8 points] Let $B = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & j \end{bmatrix}$ be some size 3x3 matrix. Assume that $\det(B) = 10$. Determine if the matrix A that appears below is invertible by computing its determinant.

$$A = \begin{bmatrix} 7a & 7b & 0 & 7c \\ 0 & 0 & 1 & 0 \\ -g & -h & 0 & -j \\ d-14a & e-14b & 0 & f-14c \end{bmatrix}$$

Hint: Compare the matrix A with B and B^T and use the basic properties of determinants.

Solution: We evaluate the determinant using the second row and then compute the resulting smaller determinant applying elementary row operations:

$$\det A = (-1)^{2+3} \det \begin{bmatrix} 7a & 7b & 7c \\ -g & -h & -j \\ d-14a & e-14b & f-14c \end{bmatrix} = (-1)^5 \det \begin{bmatrix} 7a & 7b & 7c \\ -g & -h & -j \\ d & e & f \end{bmatrix}$$
$$= (-1)^5 \cdot (-1) \cdot 7 \ \det \begin{bmatrix} a & b & c \\ g & h & j \\ d & e & f \end{bmatrix} = (-1)^5 \cdot (-1) \cdot 7 \cdot (-1) \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$
$$= (-1)^5 \cdot (-1) \cdot 7 \cdot (-1) \det B^T = (-1)^5 \cdot (-1) \cdot 7 \cdot (-1) \det B = -70.$$

Therefore the matrix A is invertible since it has a nonzero determinant.

d. [5 points] If C is a 3x3 matrix and $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are eigenvectors of C associated to the eigenvalues $\lambda = 2$ and $\lambda = -1$ respectively, find $C(v+w)$.

Solution: We have that

$$C(v+w) = Cv + Cw = 2v - w = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$$

2. [16 points] **Linear Transformations**

We denote the 2-dimensional and 3-dimensional spaces over the real numbers by $\mathbf{R^2}$ and $\mathbf{R^3}$, respectively.

a. [8 points] Let $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ be the standard basis of $\mathbf{R^3}$. Find the matrix A representing the linear transformation $T : \mathbf{R^3} \to \mathbf{R^3}$ that satisfies $T(e_1) = e_1 + 2e_2 + 3e_3$, $T(e_1 + e_2) = e_1 + 2e_2$ and $T(e_1 + e_2 + e_3) = -e_2 + e_3$.

Solution: The matrix A has the vectors $T(e_1)$, $T(e_2)$ and $T(e_3)$ as its columns. We have that:

$$T(e_1) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \quad T(e_2) = T(e_1 + e_2) - T(e_1) = \begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\-3\end{bmatrix}$$

$$T(e_3) = T(e_1 + e_2 + e_3) - T(e_1 + e_2) = \begin{bmatrix}0\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}-1\\-3\\1\end{bmatrix}$$

Therefore the matrix A is

$$A = \begin{bmatrix}1 & 0 & -1\\2 & 0 & -3\\3 & -3 & 1\end{bmatrix}$$

b. [8 points] Find the matrix D that represents the linear transformation $T : \mathbf{R^2} \to \mathbf{R^2}$ that takes any vector in the plane XY and first reflects it on the line $y = -x$ and then rotates the resulting vector by an angle of 45° counterclockwise.

Solution: The matrix representing the reflection on the line $y = -x$ is:

$$A = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

The matrix representing the rotation by an angle of 45° counterclockwise is:

$$B = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$$

Since the linear transformation represented by D consist of applying first the given reflection and then the given rotation we have that $D = BA$. Therefore

$$D = BA = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\-1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}.$$

3. [18 points] Suppose that the matrix P given below is the matrix of a random walk:

$$P = \begin{bmatrix} 1/10 & 2/10 & 3/10 \\ 3/10 & 5/10 & 7/10 \\ 6/10 & 3/10 & 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 6 & 3 & 0 \end{bmatrix}.$$

a. [4 points] What is the probability that a walker starting in location 2 is in location 1 after one time step?

Solution: The probability that a walker starting in location 2 is in location 1 after one time step is $p_{12} = 2/10 = 0.2$.

b. [7 points] What is the probability that a walker starting in location 2 is in location 1 after two time steps?

Solution: The probability that a walker starting in location 2 is in location 1 after two time steps is the entry in the first row and second column of the matrix P^2. We have that

$$P^2 = PP = \frac{1}{100}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 6 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 6 & 3 & 0 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 25 & 21 & 17 \\ 60 & 52 & 44 \\ 15 & 27 & 39 \end{bmatrix}.$$

Then the required probability is $21/100 = 0.21$.

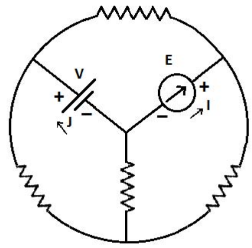
c. [7 points] If $x_0 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, what is the probability that the walker is in location 3 after two time steps?

Solution: The probability that the walker is in location 3 after two time steps is the third entry of the vector $P^2 x_0$. We have that

$$P^2 x_0 = \frac{1}{100}\frac{1}{2}\begin{bmatrix} 25 & 21 & 17 \\ 60 & 52 & 44 \\ 15 & 27 & 39 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{200}\begin{bmatrix} 38 \\ 96 \\ 66 \end{bmatrix} = .$$

Then the required probability is $\frac{66}{200} = 0.33$.

4. [20 points] Consider the resistor network in the following graph.



This resistor network has four resistors, each of them having a resistency of 1 Ω. As usual, we represent the voltage difference across the power source by V and the current across the current source by I. Take the orientation of V, E, I and J as shown in the graph.
Express J the current through the power source and E the voltage drop across the current source in terms of arbitrary values of the sources V and I.

Solution: We will use the method of *loop currents*. There are three elementary loops. Let i_1, i_2 and i_3 be the clockwise currents through the top, bottom-left and bottom-right loops. We use as unknowns i_1, i_2 , i_3 and E (and we compute $J = i_1 - i_2$ afterwards). The equations of voltage drops through the top loop, bottom-left loop and right loop are respectively

$$\begin{aligned} i_1 + E - V &= 0, \\ (i_2 - i_3) + i_2 + V &= 0, \\ i_3 + (i_3 - i_2) - E &= 0. \end{aligned}$$

The equation of obtained from the current source is

$$-i_1 + i_3 = I.$$

Putting these together we get the linear system:

$$\begin{array}{rrrrl} i_1 & & & +E & = V \\ & 2i_2 & - i_3 & & = -V \\ & - i_2 & + 2i_3 & - E & = 0 \\ -i_1 & & + i_3 & & = I \end{array}$$

We solve this system using Gaussian elimination:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & V \\ 0 & 2 & -1 & 0 & -V \\ 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & I \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{5}V - \frac{3}{5}I \\ 0 & 1 & 0 & 0 & -\frac{2}{5}V + \frac{1}{5}I \\ 0 & 0 & 1 & 0 & \frac{1}{5}V + \frac{2}{5}I \\ 0 & 0 & 0 & 1 & \frac{4}{5}V + \frac{3}{5}I \end{array}\right]$$

Then

$$i_1 = \frac{1}{5}V - \frac{3}{5}I, \quad i_2 = \frac{2}{5}V + \frac{1}{5}I, \quad i_3 = \frac{1}{5}V + \frac{2}{5}I \quad \text{and} \quad E = \frac{4}{5}V + \frac{3}{5}I.$$

Therefore, we can express J the current through the power source and E the voltage drop across the current source in terms of arbitrary values of the sources V and I as follows:

$$E = \frac{4}{5}V + \frac{3}{5}I, \qquad J = i_1 - i_2 = \frac{3}{5}V - \frac{4}{5}I.$$

5. [20 points] **Eigen-analysis**.
Solve the following exercises showing all your work.

a. [10 points] Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}$.

Solution: First we solve the equation $\det(A - \lambda I) = 0$ to find the eigenvalues of A. We have that

$$\det(A - \lambda I) = \begin{bmatrix} -2-\lambda & -1 \\ 4 & 3-\lambda \end{bmatrix} = (-2-\lambda)(3-\lambda) + 4 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Then, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -1$.
To find the eigenvalues associated to $\lambda_1 = 2$ we solve the homogeneous system $(A-2I)x = 0$. We proceed by Gaussian elimination,

$$\begin{bmatrix} -2-2 & -1 \\ 4 & 3-2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix} \to \begin{bmatrix} -4 & -1 \\ 0 & 0 \end{bmatrix}$$

Then $-4x_1 - x_2 = 0$, and hence $x_1 = -1/4x_2$. Then the space of eigenvectors corresponding to $\lambda_1 = 2$ is given by the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$, where x_2 is a free parameter.
To find the eigenvalues associated to $\lambda_2 = -1$ we solve the homogeneous system $(A + I)x = 0$. We proceed by Gaussian elimination,

$$\begin{bmatrix} -2+1 & -1 \\ 4 & 3+1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

Then $-x_1 - x_2 = 0$, and hence $x_1 = -x_2$. Then the space of eigenvectors associated to $\lambda_2 = -1$ is given by the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, where x_2 is a free parameter.

[Note: Solution to 5b) on the next page]

b. [10 points] The complex number $\lambda = 2 + i$ is an eigenvalue of the real matrix $A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 6 & 17 \\ 0 & -1 & -2 \end{bmatrix}$. Find the eigenvectors corresponding to the eigenvalue $\lambda = 2 + i$. Express all complex numbers that appear in your final answer in the form $a + b\,i$.

Solution:
To find the eigenvalues associated to $\lambda = 2 + i$ we solve the homogeneous system $(A - (2+i)I)x = 0$. We proceed by Gaussian elimination,

$$\begin{bmatrix} 2-(2+i) & 0 & -7 \\ 0 & 6-(2+i) & 17 \\ 0 & -1 & -2-(2+i) \end{bmatrix} = \begin{bmatrix} -i & 0 & -7 \\ 0 & 4-i & 17 \\ 0 & -1 & -4-i \end{bmatrix} \to \begin{bmatrix} -i & 0 & -7 \\ 0 & 4-i & 17 \\ 0 & 0 & 0 \end{bmatrix}$$

Then we get $-ix_1 - 7x_3 = 0$ and $(4-i)x_2 + 17x_3 = 0$. From the first equation we get $x_1 = 7ix_3$ and form the second we get $x_2 = -\frac{17}{4-i}x_3 = -\frac{17(4+i)}{(4-i)(4+i)}x_3 = -\frac{17(4+i)}{17}x_3 = (-4-i)x_3$.
Then the space of eigenvectors associated to $\lambda = 2 + i$ is given by the vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 7i \\ -4-i \\ 1 \end{bmatrix},$$

where x_3 is a free parameter.