Math 152 — First Midterm

Feb 7, 2012

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1. **Do not open this exam until you are told to do so.**

2. **SPECIAL INSTRUCTIONS:** No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

3. **Turn off all cell phones and pagers**, and remove all headphones.

4. This exam has 8 pages including this cover. There are 5 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

5. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

6. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

7. Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action. (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners. (b) Speaking or communicating with other candidates. (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

8. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam. You must not take any examination material from the examination room without permission of the invigilator.

9. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. [20 points] Let $\vec{v}_1$ and $\vec{v}_2$ be the vectors below

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

a. [8 points] Find the projection $\operatorname{proj}_{\vec{v}_2} \vec{v}_1$ of the vector $\vec{v}_1$ onto the vector $\vec{v}_2$.

Solution: **Solution**: Note that

$$\vec{v}_1 \cdot \vec{v}_2 = (1)(1) + (0)(1) + (1)(0) = 1,$$

$$\|\vec{v}_2\|^2 = 1^2 + 1^1 + 0^2 = 2.$$

Using the formula for projection,

$$\operatorname{proj}_{\vec{v}_2} \vec{v}_1 = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

b. [8 points] Find a nonzero vector $\vec{w}$ which is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

Solution: We can take $\vec{w} = \vec{v}_1 \times \vec{v}_2$. Therefore,

$$\vec{w} = \vec{v}_1 \times \vec{v}_2 = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{i} - \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{j} + \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{k}$$

$$= (-1)\vec{i} - (-1)]\vec{j} + (1)\vec{k} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Alternative solution: Since $\vec{w}$ is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$, we have $\vec{w} \cdot \vec{v}_1 = 0$ and $\vec{w} \cdot \vec{v}_2 = 0$. Let $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Hence, the components of $\vec{w}$ have to satisfy two equations

$$\begin{cases} w_1 + w_3 = 0 \\ w_1 + w_2 = 0 \end{cases}.$$

Any nontrivial solution to the above system gives such a $\vec{w}$, for example, take $w_1 = -1$ and $w_2 = w_3 = 1$.

c. [4 points] Find the area of the parallelogram with sides $\vec{v}_1$ and $\vec{v}_2$.

Solution: The area of the parallelogram is given by $\|\vec{v}_1 \times \vec{v}_2\|$, using the result in (b), we know

$$\|\vec{v}_1 \times \vec{v}_2\| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}.$$

2. [20 points]

    a. [12 points] Find the intersection of the plane P described by the equation $2x+3y-5z = -6$ and line L described in parametric form by

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

where t is a free parameter.

Solution: From the parametric form of the line L:

$$\begin{cases} x = 3 - t \\ y = 2 + t \\ z = 4 \end{cases} . \tag{1}$$

Substitute into the equation for the plane P and solve for the parameter t,

$$2(3 - t) + 3(2 + t) - 5(4) = -6$$

we get $t = 2$. Putting back into (1) above, the intersection point is $(1, 4, 4)$.

Alternative Solution: First we get an equation form for the line L. From the parametric form, the line L is parallel to the vector $\vec{a} = [-1, 1, 0]$. Take two linearly independent normal vectors $\vec{n}_1$ and $\vec{n}_2$ to the line L, i.e. $\vec{n}_1 \cdot \vec{a} = 0 = \vec{n}_2 \cdot \vec{a}$, e.g. $\vec{n}_1 = [1, 1, 0]$ and $\vec{n}_2 = [1, 1, 1]$ (these can be obtained by solving the equation $-x + y = 0$). Since L passes through the point $\vec{q} = [3, 2, 4]$, an equation form for L is given by the equations $\vec{n}_1 \cdot \vec{x} = \vec{n}_1 \cdot \vec{q}$ and $\vec{n}_2 \cdot \vec{x} = \vec{n}_2 \cdot \vec{q}$, i.e.

$$\begin{cases} x + y = 5 \\ x + y + z = 9 \end{cases} .$$

To find the intersection of P and L, we put all the equations together into a 3 by 3 system

$$\begin{cases} x + y = 5 \\ x + y + z = 9 \\ 2x + 3y - 5z = -6 \end{cases} .$$

Solving the system gives the unique solution $x = 1$, $y = z = 4$.

b. [8 points] When solving the linear system

$$\begin{cases} x - 3y + z = -2 \\ 2x - 5y - z = -2 \\ -3x + 10y - 6z = 8 \end{cases}$$

we get the following reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Use this to find all the solutions to the associated homogeneous system

$$\begin{cases} x - 3y + z = 0 \\ 2x - 5y - z = 0 \\ -3x + 10y - 6z = 0 \end{cases}$$

Solution: The reduced row echelon form of the associated homogeneous system is

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, z is a free parameter and x, y are basic variables. We can write the system as

$$\begin{cases} x = 8z \\ y = 3z \end{cases}.$$

Therefore, the solutions to the associated homogeneous system are

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Alternative Solution: From the reduced row echelon form, the solutions to the inhomogeneous system is given by

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Since we have $S = \vec{q} + S_h$ where S is the solution set to the inhomogeneous system, S_h is the solution set to the associated homogeneous system and $\vec{q}$ is any particular solution to the inhomogeneous system. Therefore, the solution set to the homogeneous system is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

3. [20 points]

a. [6 points] Compute

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

Solution: By definition,

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= (4 - 3) - (2 - 0) + (1 - 0)$$

$$= 1 - 2 + 1 = 0.$$

b. [6 points] Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ be the vectors below

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Are the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ linearly independent? **Explain your answer** (*Hint*: You can use the result from (a)).

Solution: No, they are not linearly independent. Note that the matrix in part (a) has columns consisting of exactly $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$, as the determinant is zero, they must be linearly dependent.

c. [8 points] Suppose that when solving a certain linear system, the row echelon form is

$$\begin{bmatrix} 1 & 2 & 3 & | & 4 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & a & | & b \end{bmatrix}$$

where a and b are some numbers. Find all possible values of a and b such that the system has *at least* one solution. **Explain your answer** (*Hint*: To help you write your answer you can consider different cases).

Solution: The system has a a solution as long as there is no row which has all zeros on the left hand side but nonzero on the right hand side, which only happens when $a = 0$ and $b \neq 0$. Therefore, the answer is

(i) $a = b = 0$: in this case, there are infinitely many solution since x_3 becomes a free parameter.

(ii) $a \neq 0$ and $b \in \mathbb{R}$: in this case, there is a unique solution.

4. [20 points] The solution set to a linear equation in 4-dimensional space is called a *hyperplane*. For example:

$$\begin{array}{lll} H_1: & \text{Solutions of} & x + y - 2z + 3w = -3 \\ H_2: & \text{Solutions of} & -2x - y + 3z - 5w = 5 \\ H_3: & \text{Solutions of} & -3x - 4y + 8z - 9w = 13 \end{array}$$

are three distinct hyperplanes. Find a parametric description of the intersection of the hyperplanes H_1, H_2 and H_3 (Hint: This intersection is a line).

Solution: We have to solve the following system

$$\left\{ \begin{array}{l} x + y - 2z + 3w = -3 \\ -2x - y + 3z - 5w = 5 \\ -3x - 4y + 8z - 9w = 13 \end{array} \right. .$$

The augmented matrix to the above system is

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & -3 \\ -2 & -1 & 3 & -5 & 5 \\ -3 & -4 & 8 & -9 & 13 \end{array} \right].$$

Add two times the first row to the second row and three times the first row to the third row, we get

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & -3 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 0 & 4 \end{array} \right].$$

Subtract the second row from the first row and add the second row to the third row, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 3 \end{array} \right].$$

Add the third row to the first row and second row, we get the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{array} \right].$$

Therefore, the system reduces to

$$\left\{ \begin{array}{l} x = 1 - 3w \\ y = 2 - 2w \\ z = 3 - w \end{array} \right. .$$

and the solution, i.e. the intersection of the hyperplanes, is given in parametric form by

$$\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; s \in \mathbb{R} \right\}.$$

5. [20 points] *True/False.* For each sentence below, circle **T** if it is true and circle **F** if it is false. You do not need to explain the answers.

    a. [4 points] If $\vec{a}$ and $\vec{b}$ are orthogonal to $\vec{c}$, then $\vec{a} + \vec{b}$ is also orthogonal to $\vec{c}$.

<div align="right">

T   **F**

</div>

> *Solution:* Using the algebraic properties of dot products, we have
> $$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = 0 + 0 = 0.$$

    b. [4 points] An equation for the line $L = \{(1, 2) + s(3, 1) | s \in \mathbb{R}\}$ is given by $2x + y = 4$.

<div align="right">

T   **F**

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> *Solution:* False. The line L is parallel to the vector $[3, 1]$ while the line given by $2x + y = 4$ is normal to $[2, 1]$, however $[3, 1] \cdot [2, 1] = 7 \neq 0$.

    c. [4 points] A system of 2 equations with 3 unknowns can have a unique solution.

<div align="right">

T   **F**

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> *Solution:* False. Since there can be at most one pivot in each row in the reduced row echelon form, the rank of its augmented matrix is at most 2, which is strictly less than the number of unknowns. In other words, there is at least one free parameter in the reduced row echelon form. Therefore, there are either no solutions or infinitely many solutions.

    d. [4 points] A system of 3 equations in 2 unknowns can have a unique solution.

<div align="right">

T   **F**

</div>

> *Solution:* True. For example, the system
> $$\begin{cases} x + y = 0 \\ x - y = 0 \\ 2x - 2y = 0 \end{cases}$$
> has a unique solution $x = y = 0$.

    e. [4 points] The point $(1, 2, 0)$ can be the unique solution to a homogeneous system.

<div align="right">

T   **F**

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> *Solution:* False. Any homogeneous system has $(0, 0, 0)$ as one of its solutions.